SYMMETRIC AND NONSYMMETRIC HALL–LITTLEWOOD POLYNOMIALS OF TYPE BC

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Abstract. Koornwinder polynomials $K_\lambda(z_1, \dots, z_n; q, t; t_0, \dots, t_3)$ are a 6-parameter BC_n-symmetric family of Laurent polynomials indexed by partitions, from which Macdonald polynomials can be recovered in suitable limits of the parameters. As in the Macdonald polynomial case, standard constructions via difference operators do not allow one to directly control these polynomials at $q = 0$. In the first part of this paper, we provide an explicit construction for these polynomials in this limit, using the defining properties of Koornwinder polynomials. Our formula is a first step in developing the analogy between Hall-Littlewood polynomials and Koornwinder polynomials at $q = 0$. Next, we provide an analogous construction for the nonsymmetric Koornwinder polynomials in the same limiting case. The method employed in this paper is a BC-type adaptation of techniques used in an earlier work of the author, which gave a combinatorial method for proving vanishing results of Rains and Vazirani at the Hall-Littlewood level. As a consequence of this work, we obtain direct arguments for the constant term evaluations and norms in both the symmetric and nonsymmetric cases.

1. Introduction

In [6], Macdonald introduced a very important family of multivariate q-orthogonal polynomials associated to a root system. These polynomials, and their connections to representation theory, combinatorics and algebra, have been well-studied and are an active area of research. For the type A root system Macdonald polynomials $P_\lambda^{(n)}(x; q, t)$ contain many well-known families of symmetric functions as special cases: for example, the Schur, Hall-Littlewood, and Jack polynomials occur at $q = t$, $q = 0$, and $t = q^\alpha, q \to 1$, respectively. The existence of the top level Macdonald polynomials was proved by exhibiting a suitable operator which has these polynomials as its eigenfunctions. A particularly important degeneration of the Macdonald polynomials is at the $q = 0$ level where one obtains zonal spherical functions on semisimple p-adic groups. In fact, Macdonald provides an explicit formula for the spherical functions of the Chevalley group $G(\mathbb{Q}_p)$ in terms of the root data for the group G [4]. In particular, this generalizes the formula for the Hall-Littlewood polynomials [5, Ch. III], which arise as zonal spherical functions for $Gl_n(\mathbb{Q}_p)$.

In [3], Koornwinder introduced a remarkable class of multivariate q-orthogonal polynomials associated to the non-reduced root system BC_n. These polynomials are a 6-parameter family of Laurent polynomials which are invariant under permuting variables and taking inverses of variables. Moreover, these polynomials reduce to the Askey-Wilson polynomials at $n = 1$, and one recovers the Macdonald polynomials by taking suitable limits of the parameters [1]. As in the Macdonald polynomial case, the existence of these polynomials was proved by using q-difference operators; these behave badly as $q \to 0$. The explicit construction for the Macdonald polynomials at $q = 0$ is due to Littlewood; in fact, Hall also provided a definition of them indirectly via the Hall algebra. Given the relationship between Macdonald and Koornwinder polynomials, a natural question is whether there exists an explicit construction for the latter polynomials at $q = 0$, thereby providing an analog of the construction of Hall-Littlewood polynomials for this family. In this work, we use the defining properties of Koornwinder polynomials to provide a closed formula at the $q = 0$ level. We then extend this technique to study the nonsymmetric Koornwinder polynomials in the $q = 0$ limit. We provide an explicit formula when these polynomials are indexed by *partitions*; we then use elements of the affine Hecke algebra of type BC to recursively obtain *all* nonsymmetric Koornwinder polynomials in this limit. A nice feature of this work is the self-contained proofs of the constant term evaluations and norm evaluations in both the symmetric and nonsymmetric cases (Theorems 2.7, 3.4 and Theorems 2.8, 3.6). We mention

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that, in the symmetric case, the constant term evaluation at the q-level is a result of Gustafson [2]. However, that approach requires $q \neq 0$, so one cannot directly apply that argument in this limiting case.

The motivation for this problem arose when the author was investigating direct proofs at the Hall-Littlewood level for the vanishing results of Rains and Vazirani [9] (note that many of those results were first conjectured in [8]). These identities are (q,t)-generalizations of restriction rules for Schur functions. More precisely, one integrates a suitably specialized Macdonald polynomial indexed by λ against a particular density; the result vanishes unless λ satisfies an explicit condition and at $q = t$ one recovers an (often known) identity about Schur functions. In [13], we demonstrate a combinatorial technique for handling the Hall-Littlewood level. This method makes use of the structure of the Hall-Littlewood polynomials as a sum over the Weyl group. Many of the results in [9] involve Koornwinder polynomials so, in order to extend the method developed in that paper, it is necessary to first find a closed formula at the $q = 0$ level. In fact, the technique used in this paper to prove that the specified polynomials are orthogonal with respect to the Koornwinder density is a natural adaptation of the ideas in [13] to the type BC_n case.

This paper has two main components: the first part deals with the symmetric theory at $q = 0$, while the second deals with the nonsymmetric theory in the same limit. The first section of each part sets up the relevant notation, reviews some background material, and defines the polynomials in question. The second section of each part consists of the main theorems and proofs, in particular we prove that these are indeed the symmetric and nonsymmetric Koornwinder polynomials at $q = 0$, respectively. As mentioned above, [9] provided vanishing results involving suitably specialized (symmetric) Koornwinder polynomials. The third section of the first part contains an application of our formula to this work: we use the construction of the Koornwinder polynomials at $q = 0$ to provide a new combinatorial proof of a result from [9].

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2. Symmetric Hall-Littlewood polynomials of type BC

2.1. **Background and Notation.** We will first review some relevant notation before introducing the polynomials that are these subject of this paper; a good reference is [5, Ch. 1].

Recall that a partition λ is a non-increasing string of positive integers $(\lambda_1, \lambda_2, \dots, \lambda_n)$, in which some of the λ_i may be zero. We call the λ_i the "parts" of λ. We write $l(\lambda) = \max\{k \geq 0 | \lambda_k \neq 0\}$ (the "length") and $|\lambda| = \sum_{i=1}^n \lambda_i$ (the "weight"). A string $\mu = (\mu_1, \dots, \mu_n)$ of integers (not necessarily non-increasing or positive) is called a composition of $|\mu| = \sum_{i=1}^n \mu_i$. We will say λ is an "even partition" if all parts of λ are even; in this case we may write $\lambda = 2\mu$ where $\mu_i = \lambda_i/2$ for all i.

We briefly recall some orderings on compositions.

Definition 2.1. Let $\leq$ denote the dominance partial ordering on compositions, i.e., $\mu \leq \lambda$ if and only if

$$\sum_{1 \leq i \leq k} \mu_i \leq \sum_{1 \leq i \leq k} \lambda_i$$

for all $k \geq 1$ (and $\mu < \lambda$ if $\mu \leq \lambda$ and $\mu \neq \lambda$). Let $\stackrel{\text{lex}}{\leq}$ denote the reverse lexicographic ordering: $\mu \stackrel{\text{lex}}{\leq} \lambda$ if and only if $\lambda = \mu$ or the first non-vanishing difference $\lambda_i - \mu_i$ is positive.

Note that $\stackrel{\text{lex}}{\leq}$ is a total ordering.

Definition 2.2. Let μ and λ be two elements of $\mathbb{Z}^n$. We will write μ^+ for the unique dominant weight in the BC_n orbit of μ (that is, the partition obtained by rearranging the absolute values of the parts of μ in non-increasing order). Then we write $\mu \prec \lambda$ if and only if either 1) $\mu^+ < \lambda$ or 2) $\mu^+ = \lambda$ and $\mu < \lambda$, and in either case $\mu \neq \lambda$.

Lemma 2.3. *Let $\mu, \lambda \in \mathbb{Z}^n$ such that $\mu \leq \lambda$. Then $\mu \stackrel{lex}{\leq} \lambda$.*

Proof. The claim is clearly true if $\mu = \lambda$, so suppose $\mu < \lambda$. If $\mu_1 < \lambda_1$, we're done; otherwise $\mu_1 = \lambda_1$ and $\mu_2 \leq \lambda_2$. Continuing this way produces an integer i in $\{1, \dots, n\}$ such that $\mu_1 = \lambda_1, \dots, \mu_{i-1} = \lambda_{i-1}$ and $\mu_i < \lambda_i$. Thus, $\mu \stackrel{\text{lex}}{<} \lambda$ as desired. $\square$

Remarks. This section will mostly deal with partitions and the dominance and reverse lexicographic orderings. Compositions, and the extended dominance ordering of the previous definition, will become relevant in the following section dealing with the nonsymmetric theory.

Let $m_i(\lambda)$ be the number of λ_j equal to i for each $i \geq 0$. Then we define:

$$v_\lambda(t;a,b;t_0,\dots,t_3) = \left(\prod_{i\geq 0}\prod_{j=1}^{m_i(\lambda)}\frac{1-t^j}{1-t}\right)\prod_{i=1}^{m_1(\lambda)}(1-t_0t_1t_2t_3t^{i-1+2m_0(\lambda)})\prod_{i=1}^{m_0(\lambda)}(1-abt^{i-1}), \tag{1}$$

and

$$v_{\lambda+}(t;t_0,\dots,t_3) = \left(\prod_{i\geq 1}\prod_{j=1}^{m_i(\lambda)}\frac{1-t^j}{1-t}\right)\prod_{i=1}^{m_1(\lambda)}(1-t_0t_1t_2t_3t^{i-1+2m_0(\lambda)}). \tag{2}$$

Note the comparison with the factors making the Hall-Littlewood polynomials monic in [5, Ch. III].

Throughout this paper, we will use

$$T = T_n = \{(z_1,\dots,z_n) : |z_1| = \cdots = |z_n| = 1\},$$
$$dT = \prod_{1\leq j\leq n}\frac{dz_j}{2\pi\sqrt{-1}z_j}$$

to denote the n-torus and Haar measure, respectively. Since many of the objects we will be dealing with are functions of n variables, we will often use the superscript (n) with z in the argument, instead of $(z_1,\dots,z_n)$. We define the q-symbol

$$(a;q) = \prod_{k\geq 0}(1-aq^k)$$

and let $(a_1,a_2,\dots,a_l;q)$ denote $(a_1;q)(a_2;q)\cdots(a_l;q)$.

We recall the symmetric Koornwinder density [3]:

$$\tilde{\Delta}_K^{(n)}(z;q,t;t_0,\dots,t_3) = \frac{(q;q)^n}{2^n n!}\prod_{1\leq i\leq n}\frac{(z_i^{\pm 2};q)}{(az_i^{\pm 1},bz_i^{\pm 1},cz_i^{\pm 1},dz_i^{\pm 1};q)}\prod_{1\leq i<j\leq n}\frac{(z_i^{\pm 1}z_j^{\pm 1};q)}{(tz_i^{\pm 1}z_j^{\pm 1};q)}.$$

Since we are interested in $q = 0$ degenerations of Koornwinder polynomials, we will be interested in the symmetric Koornwinder density in the same limiting case:

$$\begin{aligned}\tilde{\Delta}_K^{(n)}&(z;0,t;t_0,t_1,t_2,t_3)\\ &= \frac{1}{2^n n!}\prod_{1\leq i\leq n}\frac{(1-z_i^{\pm 2})}{(1-t_0z_i^{\pm 1})(1-t_1z_i^{\pm 1})(1-t_2z_i^{\pm 1})(1-t_3z_i^{\pm 1})}\prod_{1\leq i<j\leq n}\frac{(1-z_i^{\pm 1}z_j^{\pm 1})}{(1-tz_i^{\pm 1}z_j^{\pm 1})},\end{aligned} \tag{3}$$

where we write $(1-z_i^{\pm 2})$ for the product $(1-z_i^2)(1-z_i^{-2})$ and $(1-z_i^{\pm 1}z_j^{\pm 1})$ for $(1-z_iz_j)(1-z_i^{-1}z_j^{-1})(1-z_i^{-1}z_j)(1-z_iz_j^{-1})$, etc. We will write $\tilde{\Delta}_K^{(n)}(z;t;t_0,\dots,t_3)$ to denote this density.

Using this density, we let

$$N_\lambda(t;t_0,\dots,t_3) = \frac{1}{v_{\lambda+}(t)}\int_T \tilde{\Delta}_K^{(m_0(\lambda))}(z;t;t_0,\dots,t_3)dT \tag{4}$$

We note that, at the q-level, the explicit evaluation of the integral above is a famous result of Gustafson [2]. However, the arguments do not directly apply at $q = 0$. In keeping with the theme of this work, we will provide a self-contained proof of the evaluation of this integral in Theorem 2.7. This will provide an explicit formula for the quantity $N_\lambda(t;t_0,\dots,t_3)$.

For simplicity of notation, we will write $v_\lambda, v_{\lambda+}, N_\lambda, \tilde{\Delta}_K^{(n)}$, etc., when the parameters are clear from the context.

Finally, we explain some notation involving elements of the hyperoctahedral group, B_n. An element in B_n is determined by specifying a permutation $\rho \in S_n$ as well as a sign choice $\epsilon_\rho(i)$, for each $1 \leq i \leq n$. Thus, ρ acts on the subscripts of the variables, for example by

$$\rho(z_1\cdots z_n) = z_{\rho(1)}^{\epsilon_\rho(1)}\cdots z_{\rho(n)}^{\epsilon_\rho(n)}.$$

If $\rho(i)=1$, we will say that z_1 occurs in position i of ρ. We also write

$$\text{“}z_i \prec z_j\text{”}$$

if $i=\rho(i')$ and $j=\rho(j')$ for some $i'<j'$, i.e., z_i appears to the left of z_j in the permutation $z_{\rho(1)}^{\epsilon_\rho(1)}\cdots z_{\rho(n)}^{\epsilon_\rho(n)}$. We also define $\epsilon_\rho(z_i)$ to be $\epsilon_\rho(i')$ if $i=\rho(i')$, i.e., it is the exponent (±1) on z_i in $z_{\rho(1)}^{\epsilon_\rho(1)}\cdots z_{\rho(n)}^{\epsilon_\rho(n)}$.

We finally define the main objects of this section.

Definition 2.4. Let λ be a partition with $l(\lambda)\le n$ and $|t|,|t_0|,\dots,|t_3|<1$. Then $K_\lambda(z_1,\dots,z_n;t;a,b;t_0,\dots,t_3)$, indexed by λ, is defined by

$$\frac{1}{v_\lambda(t;a,b;t_0,\dots,t_3)}\sum_{w\in B_n} w\Bigg(\prod_{1\le i\le n}u_\lambda(z_i)\prod_{1\le i<j\le n}\frac{1-tz_i^{-1}z_j}{1-z_i^{-1}z_j}\frac{1-tz_i^{-1}z_j^{-1}}{1-z_i^{-1}z_j^{-1}}\Bigg), \tag{5}$$

where

$$u_\lambda(z_i)=\begin{cases}\frac{(1-az_i^{-1})(1-bz_i^{-1})}{1-z_i^{-2}} & \text{if } \lambda_i=0,\\ z_i^{\lambda_i}\frac{(1-t_0z_i^{-1})(1-t_1z_i^{-1})(1-t_2z_i^{-1})(1-t_3z_i^{-1})}{1-z_i^{-2}} & \text{if } \lambda_i>0.\end{cases}$$

Remarks. We note that the K_λ are actually independent of a,b - this is a scaling factor accounted for in v_λ. In particular, the arguments below for showing this is indeed the Koornwinder polynomial at $q=0$ work for any choice of a,b. However, we leave in arbitrary a,b (as opposed to the choice ±1) because the resulting form is useful for proving the vanishing identities.

We will also let

$$R_\lambda^{(n)}(z;t;a,b;t_0,\dots,t_3)=v_\lambda(t;a,b;t_0,\dots,t_3)K_\lambda^{(n)}(z;t;a,b;t_0,\dots,t_3), \tag{6}$$

and for $w\in B_n$, we let

$$R_{\lambda,w}^{(n)}(z;t;a,b;t_0,\dots,t_3)=w\Bigg(\prod_{1\le i\le n}u_\lambda(z_i)\prod_{1\le i<j\le n}\frac{1-tz_i^{-1}z_j}{1-z_i^{-1}z_j}\frac{1-tz_i^{-1}z_j^{-1}}{1-z_i^{-1}z_j^{-1}}\Bigg) \tag{7}$$

be the associated term in the summand. Finally, we will write $K_\lambda^{(n)}, R_\lambda^{(n)}$ and $R_{\lambda,w}^{(n)}$ when the parameters are clear from context.

Remarks. When $(t_0,t_1,t_2,t_3)=(a,b,0,0)$, we obtain

$$\begin{aligned}&K_\lambda(z_1,\dots,z_n;t;a,b;a,b,0,0)\\&\qquad=\frac{1}{v_\lambda(t)}\sum_{w\in B_n}w\Bigg(\prod_{1\le i\le n}z_i^{\lambda_i}\frac{(1-az_i^{-1})(1-bz_i^{-1})}{1-z_i^{-2}}\prod_{1\le i<j\le n}\frac{1-tz_i^{-1}z_j}{1-z_i^{-1}z_j}\frac{1-tz_i^{-1}z_j^{-1}}{1-z_i^{-1}z_j^{-1}}\Bigg),\end{aligned}$$

which gets rid of the difference in zero and nonzero parts in the univariate terms. In particular, this is Macdonald's 2-parameter family $(BC_n,B_n)=(BC_n,C_n)$ polynomials at $q=0$. We'll write $K_\lambda^{(n)}(z;t;a,b,0,0)$ in this case.

2.2. **Main Results.**

In this section, we will show that the $K_\lambda^{(n)}(z;t;a,b;t_0,\dots,t_3)$ are indeed the Koornwinder polynomials at $q=0$.

Theorem 2.5. *The function $K_\lambda^{(n)}(z;t;a,b;t_0,\dots,t_3)$ is a BC_n-symmetric Laurent polynomial (i.e., invariant under permuting variables $z_1,\dots,z_n$ and inverting variables $z_i\to z_i^{-1}$).*

Proof. Recall the fully BC_n-antisymmetric Laurent polynomials:

$$\Delta_{BC}=\prod_{1\le i\le n}z_i-z_i^{-1}\prod_{1\le i<j\le n}z_i^{-1}-z_j-z_j^{-1}+z_i=\prod_{1\le i\le n}\frac{z_i^2-1}{z_i}\prod_{1\le i<j\le n}\frac{1-z_iz_j}{z_iz_j}(z_j-z_i). \tag{8}$$

Then we have

$$K_\lambda^{(n)}(z;a,b;t_0,\dots,t_3;t)\cdot\Delta_{BC}=\frac{1}{v_\lambda(t)}\sum_{w\in B_n}\epsilon(w)w\Big(\prod_{1\le i\le n}u'_\lambda(z_i)\prod_{1\le i<j\le n}(1-tz_i^{-1}z_j^{-1})(z_i-tz_j)\Big), \tag{9}$$

where

$$u'_\lambda(z_i) = \begin{cases} z_i(1-az_i^{-1})(1-bz_i^{-1}) & \text{if } \lambda_i = 0, \\ z_i^{\lambda_i+1}(1-t_0z_i^{-1})\cdots(1-t_3z_i^{-1}) & \text{if } \lambda_i > 0. \end{cases}$$

Notice that $K_\lambda^{(n)} \cdot \Delta_{BC}$ is a BC_n-antisymmetric Laurent polynomial, so in particular Δ_{BC} divides $K_\lambda^{(n)} \cdot \Delta_{BC}$ as polynomials. Consequently, $K_\lambda^{(n)}$ is a BC_n-symmetric Laurent polynomial, as desired. ☐

Theorem 2.6. *The functions $K_\lambda^{(n)}(z;t;a,b;t_0,\dots,t_3)$ are triangular with respect to dominance ordering:*

$$K_\lambda^{(n)}(z;t;a,b;t_0,\dots,t_3) = m_\lambda + \sum_{\mu<\lambda} c_\mu^\lambda m_\mu.$$

Remarks. Here $\{m_\lambda\}_\lambda$ is the monomial basis with respect to Weyl group of type BC:

$$m_\lambda = \sum_{w\in B_n} w(z_1^{\lambda_1}\cdots z_n^{\lambda_n}).$$

Proof. We show that when $K_\lambda^{(n)}$ is expressed in the monomial basis, the top degree term in m_λ; moreover, it is monic. First note that from (8) in the previous proof, we have

$$\Delta_{BC} = m_\rho + (\text{dominated terms}),$$

where $\rho = (n\ n-1\ \cdots\ 2\ 1)$. We compute the dominating monomial in $K_\lambda^{(n)} \cdot \Delta_{BC}$; see (9) in the previous proof for the formula. Note that if $\lambda_i = 0$, we have highest degree $\lambda_i + 1$ in $u'_\lambda(z_i)$. Similarly, if $\lambda_i > 0$, we note that $\lambda_i + 1 \geq -\lambda_i + 3$ (with equality if and only if $\lambda_i = 1$) so we have highest degree $\lambda_i + 1$ in $u'_\lambda(z_i)$. Moreover,

$$\prod_{1\leq i<j\leq n}(1-tz_i^{-1}z_j^{-1})(z_i-tz_j) = \prod_{1\leq i<j\leq n}(z_i - tz_j^{-1} - tz_j + t^2z_i^{-1})$$

has highest degree term $z^{\rho-1}$. Thus, the dominating monomial in $K_\lambda^{(n)} \cdot \Delta_{BC}$ is $z^{\lambda+\rho}$, so that the dominating monomial in $K_\lambda^{(n)}$ is z^λ.

We now show that the coefficient on $z^{\lambda+\rho}$ in $R_\lambda^{(n)} \cdot \Delta_{BC}$ (see (6) for the definition of $R_\lambda^{(n)}$) is $v_\lambda(t)$, so that $K_\lambda^{(n)}$ is indeed monic. Note first that by the above argument the only contributing w are those such that (1) $z_1^{\lambda_1}\cdots z_n^{\lambda_n} = z_{w(1)}^{\lambda_1}\cdots z_{w(n)}^{\lambda_n}$ and (2) $\epsilon_w(z_i) = 1$ for all $1 \leq i \leq n - m_0(\lambda) - m_1(\lambda)$; let the set of these special permutations be denoted by $P_{\lambda,n}$. Now fix $w \in P_{\lambda,n}$, we compute the coefficient on $z_1^{\lambda_1+n}$. Using (9) and the arguments of the previous paragraph, one can check that the coefficient is:

(i) If $\lambda_1 > 1$:

$$t^{\#\{z_i \prec_w z_1\}}$$

(ii) If $\lambda_1 = 1$:

$$\begin{cases} t^{\#\{z_i\prec_w z_1\}}, & \text{if } \epsilon_w(z_1) = 1 \\ -t_0\cdots t_3(t^2)^{\#\{z_1\prec_w z_i\}}t^{\#\{z_i\prec_w z_1\}}, & \text{if } \epsilon_w(z_1) = -1 \end{cases}$$

(iii) If $\lambda_1 = 0$:

$$\begin{cases} t^{\#\{z_i\prec_w z_1\}}, & \text{if } \epsilon_w(z_1) = 1 \\ -ab(t^2)^{\#\{z_1\prec_w z_i\}}t^{\#\{z_i\prec_w z_1\}}, & \text{if } \epsilon_w(z_1) = -1 \end{cases}$$

(note that we have used the contribution of (-1) factors from $\epsilon(w)$ in $K_\lambda^{(n)} \cdot \Delta_{BC}$).

Now define the following subsets of the variables $z_1,\dots,z_n$:

$$\begin{aligned} N_{w,\lambda}^1 &= \{z_i : n - m_0(\lambda) - m_1(\lambda) < i \leq n - m_0(\lambda) \text{ and } \epsilon_w(z_i) = -1\} \\ N_{w,\lambda}^0 &= \{z_i : n - m_0(\lambda) < i \leq n \text{ and } \epsilon_w(z_i) = -1\} \\ N_{w,\lambda} &= N_{w,\lambda}^1 + N_{w,\lambda}^0. \end{aligned}$$

Finally, define the following statistics of w:

$$n(w) = |\{(i,j) : 1 \le i < j \le n \text{ and } z_j \prec_w z_i\}|$$
$$c_\lambda(w) = |\{(i,j) : 1 \le i < j \le n \text{ and } z_i \prec_w z_j \text{ and } z_i \in N_{w,\lambda}\}|.$$

Then by iterating the coefficient argument above, we get that the coefficient on $z^{\lambda+\rho}$ is given by

$$\sum_{w \in P_{\lambda,n}} t^{n(w)} t^{2c_\lambda(w)} (-t_0 \dots t_3)^{|N^1_{w,\lambda}|} (-ab)^{|N^0_{w,\lambda}|}.$$

Since $P_{\lambda,n} = B_{m_0(\lambda)} B_{m_1(\lambda)} \prod_{i \ge 2} S_{m_i(\lambda)}$, it is enough to show the following three cases:

$$\sum_{w \in S_m} t^{n(w)} = \prod_{j=1}^{m} \frac{1-t^j}{1-t} \tag{10}$$

$$\sum_{w \in B_m} t^{n(w)} t^{2c_{1^m}(w) + 2m_0(\lambda)} (-t_0 \dots t_3)^{\left|N^1_{w,1^m}\right|} = \prod_{j=1}^{m} \frac{1-t^j}{1-t} (1 - t_0 \cdots t_3 t^{j-1+2m_0(\lambda)}) \tag{11}$$

$$\sum_{w \in B_m} t^{n(w)} t^{2c_{0^m}(w)} (-ab)^{\left|N^0_{w,0^m}\right|} = \prod_{j=1}^{m} \frac{1-t^j}{1-t} (1 - abt^{j-1}). \tag{12}$$

To show (10), we note that the LHS is exactly enumerated by the terms of

$$(1 + t + t^2 + \cdots + t^{m-1})(1 + t + t^2 + \cdots + t^{m-2}) \cdots (1+t)(1),$$

which is equal to the RHS. Also refer to [5, Ch. III, proof of (1.2) and (1.3)]. We now show (11); (12) is analogous. One can verify that the LHS of (11) is exactly enumerated by the terms of

$$\prod_{k=1}^{m} \Big[\sum_{i=1}^{k} \big(t^{i-1} + t^{i-1} (t^2)^{m_0(\lambda)+k-i} (-t_0 \cdots t_3) \big) \Big]. \tag{13}$$

But we also have

$$\begin{aligned}\sum_{i=1}^{k} \Big(t^{i-1} + t^{i-1} (t^2)^{m_0(\lambda)+k-i} (-t_0 \cdots t_3) \Big) &= \sum_{i=1}^{k} \big(t^{i-1} - t_0 \cdots t_3 t^{k+2m_0(\lambda)-1} t^{k-i} \big) \\ &= (1 - t_0 \cdots t_3 t^{k+2m_0(\lambda)-1})(1 + t + \cdots t^{k-1}) = (1 - t_0 \cdots t_3 t^{k+2m_0(\lambda)-1}) \frac{1-t^k}{1-t};\end{aligned}$$

substituting this into (13) gives the RHS of (11) as desired.

Multiplying these functions together for each distinct part i of λ (put $m = m_i(\lambda)$ in (10), (11), and (12), depending on whether $i \ge 2, i = 1,$ or $i = 0$, respectively), and using (1) shows that the coefficient on $z^{\lambda+\rho}$ in $R^{(n)}_\lambda \cdot \Delta_{BC}$ is indeed $v_\lambda(t)$, as desired. □

We will now provide a direct proof of Gustafson's formula in the limit $q = 0$, see [2].

Theorem 2.7. *We have the following constant term evaluation in the symmetric case*

$$\begin{aligned}\int_T \tilde{\Delta}^{(n)}_K (z; t; a, b, c, d) dT = \prod_{i=0}^{n-1} &\frac{1}{(1-t^i ac)(1-t^i bc)(1-t^i cd)(1-t^i ad)(1-t^i bd)(1-t^i ab)} \\ &\times \prod_{j=0}^{n-1} (1 - t^{2n-2-j} abcd) \prod_{j=1}^{n} \frac{1-t}{1-t^j}.\end{aligned}$$

Proof. Note first that by Theorem 2.6, $K_{0^n}^{(n)}(z;t;a,b,0,0)=1$. So in particular, we have

$$\begin{aligned}\int_T \tilde{\Delta}_K^{(n)}(z;t;a,b,c,d)dT &= \int_T K_{0^n}^{(n)}(z;t;a,b,0,0)\tilde{\Delta}_K^{(n)}(z;t;a,b,c,d)dT\\ &= \frac{1}{v_{0^n}(t;a,b,0,0)}\sum_{w\in B_n}\int_T R_{0^n,w}^{(n)}(z;t;a,b,0,0)\tilde{\Delta}_K^{(n)}(z;t;a,b,c,d)dT\\ &= \frac{2^n n!}{v_{0^n}(t;a,b,0,0)}\int_T R_{0^n,\mathrm{id}}^{(n)}(z;t;a,b,0,0)\tilde{\Delta}_K^{(n)}(z;t;a,b,c,d)dT,\end{aligned}$$

where the last equality follows by symmetry of the integrand. But now using (7), one notes that

$$\begin{aligned}&2^n n! R_{0^n,\mathrm{id}}^{(n)}(z;t;a,b,0,0)\tilde{\Delta}_K^{(n)}(z;t;a,b,c,d)\\ &\quad= \prod_{1\le i\le n}\frac{(1-z_i^2)}{(1-az_i)(1-bz_i)(1-cz_i)(1-dz_i)(1-cz_i^{-1})(1-dz_i^{-1})}\prod_{1\le i<j\le n}\frac{(1-z_iz_j^{\pm1})}{(1-tz_iz_j^{\pm1})}.\end{aligned}$$

We will denote the right-hand side of the above equation by $\Delta_K^{(n)}(z;t;a,b,c,d)$.

We will now prove that

$$\int_T \Delta_K^{(n)}(z;t;a,b,c,d)dT = \prod_{i=0}^{n-1}\frac{1}{(1-t^iac)(1-t^ibc)(1-t^icd)(1-t^iad)(1-t^ibd)}\prod_{j=n-1}^{2n-2}(1-t^jabcd). \tag{14}$$

For facility of notation, we will put $I_n(z;t;a,b;c,d)=\int_T \Delta_K^{(n)}(z;t;a,b,c,d)dT$. We will prove (14) through the following two claims.

Claim 1: We have

$$\begin{aligned}I_n(z;t;a,b;c,d) &= \frac{c}{(1-ac)(1-bc)(1-dc)(c-d)}I_{n-1}(z;t;a,b;tc,d)\\ &\quad+\frac{d}{(1-ad)(1-bd)(1-cd)(d-c)}I_{n-1}(z;t;a,b;c,td),\end{aligned} \tag{15}$$

with initial conditions $I_0(z;t;a,b;c,d)=1$ and

$$I_1(z;t;a,b;c,d)=\frac{1-abcd}{(1-ac)(1-bc)(1-cd)(1-ad)(1-bd)}.$$

To prove the first claim, we note that

$$\begin{aligned}&I_n(z;t;a,b;c,d)\\ &= \int\prod_{1\le i\le n}\frac{z_i(1-z_i^2)}{(1-az_i)(1-bz_i)(1-cz_i)(1-dz_i)(z_i-c)(z_i-d)}\prod_{1\le i<j\le n}\frac{(z_j-z_i)(1-z_iz_j)}{(z_j-tz_i)(1-tz_iz_j)}\prod_{j=1}^n\frac{dz_j}{2\pi\sqrt{-1}}.\end{aligned}$$

We may now hold the variables $z_2,\dots,z_n$ fixed and integrate with respect to z_1. There are simple poles at $z_1=c$ and $z_1=d$, so by the Residue Theorem, it will be the sum of residues at these poles. Consider the residue at $z_1=c$:

$$\begin{aligned}&\int\prod_{2\le i\le n}\frac{z_i(1-z_i^2)}{(1-az_i)(1-bz_i)(1-cz_i)(1-dz_i)(z_i-c)(z_i-d)}\prod_{2\le i<j\le n}\frac{(z_j-z_i)(1-z_iz_j)}{(z_j-tz_i)(1-tz_iz_j)}\\ &\qquad\times\frac{c}{(1-ac)(1-bc)(1-cd)(c-d)}\prod_{1<j\le n}\frac{(z_j-c)(1-cz_j)}{(z_j-tc)(1-tcz_j)}\prod_{j=2}^n\frac{dz_j}{2\pi\sqrt{-1}}\\ &= C_1\int_{T_{n-1}}\prod_{2\le i\le n}\frac{z_i(1-z_i^2)}{(1-az_i)(1-bz_i)(1-tcz_i)(1-dz_i)(z_i-tc)(z_i-d)}\prod_{2\le i<j\le n}\frac{(z_j-z_i)(1-z_iz_j)}{(z_j-tz_i)(1-tz_iz_j)}dT\end{aligned}$$

where $C_1=\frac{c}{(1-ac)(1-bc)(1-cd)(c-d)}$. By renumbering the variables $(z_2,\dots,z_n)$ by $(z_1,\dots,z_{n-1})$, one sees that this is exactly $C_1I_{n-1}(z;t;a,b;tc,d)$. An analogous argument applies for the residue at $z_1=d$; this produces the second term $C_2I_{n-1}(z;t;a,b;c,td)$, where $C_2=\frac{d}{(1-ad)(1-bd)(1-cd)(d-c)}$.

To obtain the result at $n = 1$, one uses the above argument in this special case along with some algebraic manipulation. In particular, the computation of the sum of residues is as follows

$$\frac{(1-c^2)c}{(1-ac)(1-bc)(1-c^2)(1-dc)(c-d)} + \frac{(1-d^2)d}{(1-ad)(1-bd)(1-cd)(1-d^2)(d-c)}$$
$$= \frac{1}{(1-ac)(1-bc)(1-cd)(1-ad)(1-bd)}\left[\frac{c(1-ad)(1-bd)}{c-d} + \frac{d(1-ac)(1-bc)}{d-c}\right]$$
$$= \frac{1-abcd}{(1-ac)(1-bc)(1-cd)(1-ad)(1-bd)},$$

as desired. This proves the first claim.

Claim 2: We have the following solution to (15)

$$I_n(z;t;a,b;c,d) = \prod_{i=0}^{n-1} \frac{1}{(1-t^iac)(1-t^ibc)(1-t^icd)(1-t^iad)(1-t^ibd)} \prod_{j=n-1}^{2n-2}(1-t^jabcd).$$

We prove the second claim. One can first check that $n = 0, 1$ satisfies the initial conditions of (15). Then for $n \geq 2$, we have

$$\frac{c}{(1-ac)(1-bc)(1-dc)(c-d)} I_{n-1}(z;t;a,b;tc,d) + \frac{d}{(1-ad)(1-bd)(1-cd)(d-c)} I_{n-1}(z;t;a,b;c,td)$$
$$= \frac{c\prod_{j=n-2}^{2n-4} 1-t^{j+1}abcd}{(1-ac)(1-bc)(1-dc)(c-d)} \prod_{i=0}^{n-2} \frac{1}{(1-t^{i+1}ac)(1-t^{i+1}bc)(1-t^{i+1}cd)(1-t^iad)(1-t^ibd)}$$
$$+ \frac{d\prod_{j=n-2}^{2n-4} 1-t^{j+1}abcd}{(1-ad)(1-bd)(1-cd)(d-c)} \prod_{i=0}^{n-2} \frac{1}{(1-t^iac)(1-t^ibc)(1-t^{i+1}cd)(1-t^{i+1}ad)(1-t^{i+1}bd)}$$
$$= \left[\frac{c(1-t^{n-1}ad)(1-t^{n-1}bd)}{c-d} + \frac{d(1-t^{n-1}ac)(1-t^{n-1}bc)}{d-c}\right]$$
$$\times \prod_{i=0}^{n-1} \frac{1}{(1-t^iac)(1-t^ibc)(1-t^icd)(1-t^iad)(1-t^ibd)} \prod_{j=n-2}^{2n-4}(1-t^{j+1}abcd)$$

But now note the following identity for the sum inside the parentheses:

$$\frac{c(1-t^{n-1}ad)(1-t^{n-1}bd)}{c-d} + \frac{d(1-t^{n-1}ac)(1-t^{n-1}bc)}{d-c}$$
$$= \frac{c(1-t^{n-1}ad)(1-t^{n-1}bd) - d(1-t^{n-1}ac)(1-t^{n-1}bc)}{c-d}$$
$$= \frac{c-d+t^{2(n-1)}abcd^2 - t^{2(n-1)}abc^2d}{c-d} = 1-t^{2(n-1)}abcd,$$

so the above finally becomes

$$\prod_{i=0}^{n-1} \frac{1}{(1-t^iac)(1-t^ibc)(1-t^icd)(1-t^iad)(1-t^ibd)} \prod_{j=n-1}^{2(n-1)}(1-t^jabcd) = I_n(z;t;a,b;c,d),$$

which proves (14).

Thus, putting this together we have

$$\int_T \tilde{\Delta}_K^{(n)}(z;t;a,b,c,d)dT = \frac{1}{v_{0^n}(t;a,b,0,0)} \int_T \Delta_K^{(n)}(z;t;a,b,c,d)dT$$
$$= \prod_{i=0}^{n-1} \frac{1}{(1-t^iac)(1-t^ibc)(1-t^icd)(1-t^iad)(1-t^ibd)} \prod_{j=n-1}^{2n-2}(1-t^jabcd) \times \prod_{j=1}^{n}\frac{1-t}{1-t^j}\prod_{i=1}^{n}\frac{1}{1-abt^{i-1}}$$
$$= \prod_{i=0}^{n-1} \frac{1}{(1-t^iac)(1-t^ibc)(1-t^icd)(1-t^iad)(1-t^ibd)(1-t^iab)} \prod_{j=0}^{n-1}(1-t^{2n-2-j}abcd)\prod_{j=1}^{n}\frac{1-t}{1-t^j},$$

where we have used Theorem 3.4 and (1).

□

We note that the quantity $\Delta_K^{(n)}(z;t;a,b,c,d)$ which appears in the proof of Theorem 2.7 is actually the $q=0$ limit of the nonsymmetric Koornwinder density (see [9] for example); the nonsymmetric theory is investigated in the next section.

Theorem 2.8. *The family of polynomials $\{K_\lambda^{(n)}(z;t;a,b;t_0,\dots,t_3)\}_\lambda$ satisfy the following orthogonality result:*

$$\int_T K_\lambda(z_1,\dots,z_n;t;a,b;t_0,\dots,t_3)K_\mu(z_1,\dots,z_n;t;a,b;t_0,\dots,t_3)\tilde{\Delta}_K^{(n)}(z;t;t_0,\dots,t_3)dT = N_\lambda(t;t_0,\dots,t_3)\delta_{\lambda\mu}$$

(refer to (3) and (4) for the definitions of $\tilde{\Delta}_K^{(n)}$ and N_λ, respectively; also see Theorem 2.7).

Proof. By symmetry of λ,μ, we may restrict to the case where $\lambda \overset{\text{lex}}{\geq} \mu$. We assume $\lambda_1 > 0$, so we need not consider the case $\lambda = \mu = 0^n$; these assumptions hold throughout the proof. By definition of $K_\lambda^{(n)}(z;t;a,b;t_0,\dots,t_3)$ as a sum over B_n, the above integral is equal to

$$\sum_{w,\rho\in B_n}\int_T K_{\lambda,w}^{(n)}(z;t;a,b;t_0,\dots,t_3)K_{\mu,\rho}^{(n)}(z;t;a,b;t_0,\dots,t_3)\tilde{\Delta}_K^{(n)}(z;t;t_0,\dots,t_3)dT.$$

Consider an arbitrary term in this sum over $B_n \times B_n$ indexed by (w,ρ). Note that using a change of variables in the integral and inverting variables (which preserves the integral), we may assume w is the identity permutation, and all sign choices are 1 (and ρ is arbitrary). That is, we have:

$$\int_T K_\lambda(z_1,\dots,z_n;t;a,b;t_0,\dots,t_3)K_\mu(z_1,\dots,z_n;t;a,b;t_0,\dots,t_3)\tilde{\Delta}_K^{(n)}(z;t;t_0,\dots,t_3)dT$$
$$= 2^n n! \sum_{\rho\in B_n}\int_T K_{\lambda,\mathrm{id}}^{(n)}(z;t;a,b;t_0,\dots,t_3)K_{\mu,\rho}^{(n)}(z;t;a,b;t_0,\dots,t_3)\tilde{\Delta}_K^{(n)}(z;t;t_0,\dots,t_3)dT$$
$$= 2^n n! \frac{1}{v_\lambda(t)v_\mu(t)} \sum_{\rho\in B_n}\int_T R_{\lambda,\mathrm{id}}^{(n)}(z;t;a,b;t_0,\dots,t_3)R_{\mu,\rho}^{(n)}(z;t;a,b;t_0,\dots,t_3)\tilde{\Delta}_K^{(n)}(z;t;t_0,\dots,t_3)dT,$$

where $R_\lambda^{(n)}$ is as defined in (6).

We study an arbitrary term in this sum. In particular, we give an iterative formula that shows that each of these terms vanishes unless $\lambda = \mu$.

Claim 2.8.1. *Fix an arbitrary $\rho \in B_n$ and let $\rho(i) = 1$ for some $1 \leq i \leq n$. Then we have the following formula:*

$$2^n n! \int_T R_{\lambda,id}^{(n)}(z;t;a,b;t_0,\dots,t_3)R_{\mu,\rho}^{(n)}(z;t;a,b;t_0,\dots,t_3)\tilde{\Delta}_K^{(n)}dT$$
$$= \begin{cases} t^{i-1}2^{n-1}(n-1)!\int R_{\widehat{\lambda},\widehat{id}}^{(n-1)}R_{\widehat{\mu},\widehat{\rho}}^{(n-1)}\tilde{\Delta}_K^{(n-1)}dT & \text{if } \mu_i=\lambda_1 \text{ and } \epsilon_\rho(z_1)=-1,\\ t^{i-1}(t^2)^{m_0(\mu)+m_1(\mu)-i}(-t_0\cdots t_3)2^{n-1}(n-1)!\int R_{\widehat{\lambda},\widehat{id}}^{(n-1)}R_{\widehat{\mu},\widehat{\rho}}^{(n-1)}\tilde{\Delta}_K^{(n-1)}dT & \text{if } \mu_i=\lambda_1=1\\ & \text{and } \epsilon_\rho(z_1)=1,\\ 0 & \text{otherwise.}\end{cases}$$

where $\widehat{\lambda}$ and $\widehat{\mu}$ are the partitions λ and μ with parts λ_1 and μ_i deleted (respectively), and $\widehat{id}$ and $\widehat{\rho}$ are the permutations id and ρ with z_1 deleted (respectively) and signs preserved.

To prove the claim, we integrate with respect to z_1 in the iterated integral, using the definition of $R^{(n)}_{\lambda,\mathrm{id}}, R^{(n)}_{\mu,\rho}$ and $\tilde{\Delta}^{(n)}_K$.

First suppose $\mu_i > 0$. The univariate terms in z_1 are:

$$\begin{aligned}&z_1^{\lambda_1}\frac{(1-t_0z_1^{-1})\cdots(1-t_3z_1^{-1})}{(1-z_1^{-2})}z_1^{\mu_i}\frac{(1-t_0z_1^{-1})\cdots(1-t_3z_1^{-1})}{(1-z_1^{-2})}\frac{(1-z_1^{\pm 2})}{(1-t_0z_1^{\pm 1})\cdots(1-t_3z_1^{\pm 1})}\\ &= z_1^{\lambda_1+\mu_i}\frac{(-z_1^2)(1-t_0z_1^{-1})\cdots(1-t_3z_1^{-1})}{(1-t_0z_1)\cdots(1-t_3z_1)}\end{aligned}$$

if $\epsilon_\rho(z_1)=1$, and

$$\begin{aligned}&z_1^{\lambda_1}\frac{(1-t_0z_1^{-1})\cdots(1-t_3z_1^{-1})}{(1-z_1^{-2})}z_1^{-\mu_i}\frac{(1-t_0z_1)\cdots(1-t_3z_1)}{(1-z_1^{2})}\frac{(1-z_1^{\pm 2})}{(1-t_0z_1^{\pm 1})\cdots(1-t_3z_1^{\pm 1})}\\ &= z_1^{\lambda_1-\mu_i}\end{aligned}$$

if $\epsilon_\rho(z_1)=-1$.

Now suppose $\mu_i = 0$. The univariate terms in z_1 are:

$$\begin{aligned}&z_1^{\lambda_1}\frac{(1-t_0z_1^{-1})\cdots(1-t_3z_1^{-1})}{(1-z_1^{-2})}\frac{(1-az_1^{-1})(1-bz_1^{-1})}{(1-z_1^{-2})}\frac{(1-z_1^{\pm 2})}{(1-t_0z_1^{\pm 1})\cdots(1-t_3z_1^{\pm 1})}\\ &= z_1^{\lambda_1}\frac{(-z_1^2)(1-az_1^{-1})(1-bz_1^{-1})}{(1-t_0z_1)\cdots(1-t_3z_1)}\end{aligned}$$

if $\epsilon_\rho(z_1)=1$, and

$$\begin{aligned}&z_1^{\lambda_1}\frac{(1-t_0z_1^{-1})\cdots(1-t_3z_1^{-1})}{(1-z_1^{-2})}\frac{(1-az_1)(1-bz_1)}{(1-z_1^{2})}\frac{(1-z_1^{\pm 2})}{(1-t_0z_1^{\pm 1})\cdots(1-t_3z_1^{\pm 1})}\\ &= z_1^{\lambda_1}\frac{(1-az_1)(1-bz_1)}{(1-t_0z_1)\cdots(1-t_3z_1)}\end{aligned}$$

if $\epsilon_\rho(z_1)=-1$.

Notice that for the cross terms in z_1 (those involving z_j for $j\neq 1$), we have

$$\prod_{j>1}\frac{1-tz_1^{-1}z_j^{-1}}{1-z_1^{-1}z_j^{-1}}\frac{1-tz_1^{-1}z_j}{1-z_1^{-1}z_j}\times\prod_{j>1}\frac{1-z_1^{\pm 1}z_j^{\pm 1}}{1-tz_1^{\pm 1}z_j^{\pm 1}}$$

from the corresponding terms in z_1 of $R_{\lambda,\mathrm{id}}$ and the density. Combining this with the cross terms of $R_{\mu,\rho}$ in z_1 (and taking into account the various sign possibilities for ρ), we obtain

$$\prod_{\substack{z_i\prec_\rho z_1\\ \text{sign } 1 \text{ for } z_i}}\frac{t-z_1z_i}{1-tz_1z_i}\prod_{\substack{z_i\prec_\rho z_1\\ \text{sign } -1 \text{ for } z_i}}\frac{t-z_1z_i^{-1}}{1-tz_1z_i^{-1}}\prod_{z_1\prec_\rho z_j}\frac{(t-z_1z_j^{-1})(t-z_1z_j)}{(1-tz_1z_j^{-1})(1-tz_1z_j)}$$

if $\epsilon_\rho(z_1)=1$, and

$$\prod_{\substack{z_i\prec_\rho z_1\\ \text{sign } 1 \text{ for } z_i}}\frac{t-z_1z_i}{1-tz_1z_i}\prod_{\substack{z_i\prec_\rho z_1\\ \text{sign } -1 \text{ for } z_i}}\frac{t-z_1z_i^{-1}}{1-tz_1z_i^{-1}}$$

if $\epsilon_\rho(z_1)=-1$.

Thus, the integral in z_1 is:

$$\begin{cases} \int_{T_1} z_1^{\lambda_1+\mu_i} \frac{(-z_1^2)(1-t_0z_1^{-1})\cdots(1-t_3z_1^{-1})}{(1-t_0z_1)\cdots(1-t_3z_1)} \cdot \\ \prod_{\substack{z_k\prec_\rho z_1\\ \epsilon_\rho(z_k)=1}} \frac{t-z_1z_k}{1-tz_1z_k} \prod_{\substack{z_k\prec_\rho z_1\\ \epsilon_\rho(z_k)=-1}} \frac{t-z_1z_k^{-1}}{1-tz_1z_k^{-1}} \prod_{z_1\prec_\rho z_j} \frac{(t-z_1z_j^{-1})(t-z_1z_j)}{(1-tz_1z_j^{-1})(1-tz_1z_j)} dT & \text{if } \mu_i>0 \text{ and } \epsilon_\rho(z_1)=1, \\ \int_{T_1} z_1^{\lambda_1} \frac{(-z_1^2)(1-az_1^{-1})(1-bz_1^{-1})}{(1-t_0z_1)\cdots(1-t_3z_1)} \cdot \\ \prod_{\substack{z_k\prec_\rho z_1\\ \epsilon_\rho(z_k)=1}} \frac{t-z_1z_k}{1-tz_1z_k} \prod_{\substack{z_k\prec_\rho z_1\\ \epsilon_\rho(z_k)=-1}} \frac{t-z_1z_k^{-1}}{1-tz_1z_k^{-1}} \prod_{z_1\prec_\rho z_j} \frac{(t-z_1z_j^{-1})(t-z_1z_j)}{(1-tz_1z_j^{-1})(1-tz_1z_j)} dT & \text{if } \mu_i=0 \text{ and } \epsilon_\rho(z_1)=1, \\ \int_{T_1} z_1^{\lambda_1-\mu_i} \prod_{\substack{z_k\prec_\rho z_1\\ \epsilon_\rho(z_k)=1}} \frac{t-z_1z_k}{1-tz_1z_k} \prod_{\substack{z_k\prec_\rho z_1\\ \epsilon_\rho(z_k)=-1}} \frac{t-z_1z_k^{-1}}{1-tz_1z_k^{-1}} dT & \text{if } \mu_i>0 \text{ and } \epsilon_\rho(z_1)=-1, \\ \int_{T_1} z_1^{\lambda_1} \frac{(1-az_1)(1-bz_1)}{(1-t_0z_1)\cdots(1-t_3z_1)} \prod_{\substack{z_k\prec_\rho z_1\\ \epsilon_\rho(z_k)=1}} \frac{t-z_1z_k}{1-tz_1z_k} \prod_{\substack{z_k\prec_\rho z_1\\ \epsilon_\rho(z_k)=-1}} \frac{t-z_1z_k^{-1}}{1-tz_1z_k^{-1}} dT & \text{if } \mu_i=0 \text{ and } \epsilon_\rho(z_1)=-1. \end{cases}$$

In particular, the first integral vanishes unless $\lambda_1 = \mu_i = 1$; the second integral always vanishes; the third integral vanishes unless $\lambda_1 = \mu_i$; the fourth integral always vanishes. Thus, we obtain the vanishing conditions of the claim. To obtain the nonzero values, use the residue theorem and evaluate at the simple pole $z_1 = 0$ in the cases $\lambda_1 = \mu_i = 1$ and $\lambda_1 = \mu_i$. Finally, combine with the original integrand involving terms in $z_2, \ldots, z_n$ to obtain the result of the claim.

Note that in particular the claim implies that if $\lambda \neq \mu$, each term vanishes and consequently the total integral is zero. This proves the vanishing part of the orthogonality statement.

Next, we compute the norm when $\lambda = \mu$. The claim shows that only certain $\rho \in B_n$ give nonvanishing term integrals. Such permutations must satisfy

$$z_1^{\lambda_1} \cdots z_n^{\lambda_n} z_{\rho(1)}^{-\lambda_1} \cdots z_{\rho(n)}^{-\lambda_n} = 1$$

and $\epsilon_\rho(z_i) = -1$ for all $1 \leq i \leq n - m_0(\lambda) - m_1(\lambda)$. For simplicity of notation, define $B_{\lambda,n}$ to be the set of such permutations $\rho \in B_n$. Then we have:

$$\begin{aligned} \int_T K_\lambda^{(n)}(z;t;a,b;t_0,\ldots,t_3) K_\lambda^{(n)}(z;t;a,b;t_0,\ldots,t_3) \tilde{\Delta}_K^{(n)} dT &= \frac{2^n n!}{v_\lambda(t)^2} \sum_{\rho\in B_n} \int_T R_{\lambda,\mathrm{id}}^{(n)} R_{\lambda,\rho}^{(n)} \tilde{\Delta}_K^{(n)} dT \\ &= \frac{2^n n!}{v_\lambda(t)^2} \sum_{\rho\in B_{\lambda,n}} \int_T R_{\lambda,\mathrm{id}}^{(n)} R_{\lambda,\rho}^{(n)} \tilde{\Delta}_K^{(n)} dT, \end{aligned}$$

since only these permutations give nonvanishing terms.

Then, using the formula of the Claim, we have

$$\begin{aligned} &2^n n! \sum_{\rho\in B_{\lambda,n}} \int_T R_{\lambda,\mathrm{id}}^{(n)} R_{\lambda,\rho}^{(n)} \tilde{\Delta}_K^{(n)} dT \\ &= \begin{cases} C_1 \times 2^{n-m_{\lambda_1}(\lambda)} (n-m_{\lambda_1}(\lambda))! \sum_{\rho\in B_{\tilde{\lambda},n-m_{\lambda_1}(\lambda)}} \int_T R_{\tilde{\lambda},\mathrm{id}}^{(n-m_{\lambda_1}(\lambda))} R_{\tilde{\lambda},\rho}^{(n-m_{\lambda_1}(\lambda))} \tilde{\Delta}_K^{(n-m_{\lambda_1}(\lambda))} dT & \text{if } \lambda_1 > 1, \\ C_2 \times 2^{n-m_{\lambda_1}(\lambda)} (n-m_{\lambda_1}(\lambda))! \sum_{\rho\in B_{\tilde{\lambda},n-m_{\lambda_1}(\lambda)}} \int_T R_{\tilde{\lambda},\mathrm{id}}^{(n-m_{\lambda_1}(\lambda))} R_{\tilde{\lambda},\rho}^{(n-m_{\lambda_1}(\lambda))} \tilde{\Delta}_K^{(n-m_{\lambda_1}(\lambda))} dT & \text{if } \lambda_1 = 1, \end{cases} \end{aligned}$$

where

$$C_1 = \prod_{k=1}^{m_{\lambda_1}(\lambda)} \Big(\sum_{i=1}^{k} t^{i-1} \Big)$$
$$C_2 = \prod_{k=1}^{m_1(\lambda)} \Big[\sum_{i=1}^{k} \big(t^{i-1} + t^{i-1}(t^2)^{m_0(\lambda)+k-i}(-t_0t_1t_2t_3) \big) \Big]$$

and $\tilde{\lambda}$ is the partition λ with all $m_{\lambda_1}(\lambda)$ occurrences of λ_1 deleted. Iterating this argument gives that

$$\begin{aligned} &2^n n! \sum_{\rho \in B_{\lambda,n}} \int_T R^{(n)}_{\lambda,\mathrm{id}} R^{(n)}_{\lambda,\rho} \tilde{\Delta}^{(n)}_K dT \\ &= \Bigg(\prod_{j>1} \prod_{k=1}^{m_j(\lambda)} \Big(\sum_{i=1}^{k} t^{i-1} \Big) \Bigg) \Bigg(\prod_{k=1}^{m_1(\lambda)} \sum_{i=1}^{k} \Big(t^{i-1} + t^{i-1}(t^2)^{m_0(\lambda)+k-i}(-t_0 \cdots t_3) \Big) \Bigg) \\ &\qquad \times 2^{m_0(\lambda)} m_0(\lambda)! \sum_{\rho \in B_{m_0(\lambda)}} \int_T R^{(m_0(\lambda))}_{0^{m_0(\lambda)},\mathrm{id}} R^{(m_0(\lambda))}_{0^{m_0(\lambda)},\rho} \tilde{\Delta}^{(m_0(\lambda))}_K dT; \end{aligned}$$

note that the expression on the final line is exactly $\int_T {R^{(m_0(\lambda))}_{0^{m_0(\lambda)}}}^2 \tilde{\Delta}^{(m_0(\lambda))}_K dT$.

Thus,

$$\begin{aligned} &\frac{2^n n!}{v_\lambda(t)^2} \sum_{\rho \in B_{\lambda,n}} \int_T R^{(n)}_{\lambda,\mathrm{id}} R^{(n)}_{\lambda,\rho} \tilde{\Delta}^{(n)}_K dT \\ &= \frac{1}{v_{\lambda+}(t)^2} \Bigg(\prod_{j>1} \prod_{k=1}^{m_j(\lambda)} \Big(\sum_{i=1}^{k} t^{i-1} \Big) \Bigg) \Bigg(\prod_{k=1}^{m_1(\lambda)} \sum_{i=1}^{k} \Big(t^{i-1} + t^{i-1}(t^2)^{m_0(\lambda)+k-i}(-t_0 \cdots t_3) \Big) \Bigg) \\ &\qquad \times \frac{1}{v_{0^{m_0(\lambda)}}(t)^2} \int_T {R^{(m_0(\lambda))}_{0^{m_0(\lambda)}}}^2 \tilde{\Delta}^{(m_0(\lambda))}_K dT, \end{aligned}$$

since by (1) and (2) we have $v_{\lambda+}(t) \cdot v_{0^{m_0(\lambda)}}(t) = v_\lambda(t)$. Now using

$$\prod_{k=1}^{m_j(\lambda)} \Big(\sum_{i=1}^{k} t^{i-1} \Big) = \prod_{k=1}^{m_j(\lambda)} \frac{1-t^k}{1-t}$$

and

$$\begin{aligned} \sum_{i=1}^{k} \Big(t^{i-1} + t^{i-1}(t^2)^{m_0(\lambda)+k-i}(-t_0 \cdots t_3) \Big) &= \sum_{i=1}^{k} \big(t^{i-1} - t_0 \cdots t_3 t^{k+2m_0(\lambda)-1} t^{k-i} \big) \\ &= (1 - t_0 \cdots t_3 t^{k+2m_0(\lambda)-1})(1 + t + \cdots t^{k-1}) \\ &= (1 - t_0 \cdots t_3 t^{k+2m_0(\lambda)-1}) \frac{1-t^k}{1-t}, \end{aligned}$$

the above expression can be simplified to

$$\begin{aligned} \frac{1}{v_{\lambda+}(t)^2} \Big(\prod_{j \geq 1} \prod_{k=1}^{m_j(\lambda)} \frac{1-t^k}{1-t} \Big) \prod_{k=1}^{m_1(\lambda)} (1 - t_0 \cdots t_3 t^{k+2m_0(\lambda)-1}) \int_T {K^{(m_0(\lambda))}_{0^{m_0(\lambda)}}}^2 \tilde{\Delta}^{(m_0(\lambda))}_K dT &= \frac{1}{v_{\lambda+}(t)} \int_T \tilde{\Delta}^{(m_0(\lambda))}_K dT \\ &= N_\lambda(t; t_0, \ldots, t_3) \end{aligned}$$

since $K^{(m_0(\lambda))}_{0^{m_0(\lambda)}} = 1$, by Theorem 2.6. Note that, by Theorem 2.7, there is an explicit evaluation for this norm. $\square$

2.3. **Application.** In this section, we use the closed formula (5) for the Koornwinder polynomials at $q=0$ to prove a result from [9] in this special case. The idea is the same as in [13]: we use the structure of $K_\lambda^{(n)}$ as a sum over the Weyl group and the symmetry of the integral to restrict to one particular term. We obtain an explicit formula for the integral of this particular term by integrating with respect to one variable (holding the others fixed) and then proceeding by induction.

Theorem 2.9. [9, Theorem 4.10] *For partitions λ with $l(\lambda)\le n$, the integral*

$$\int_T K_\lambda(z_1,\dots,z_n;t^2;a,b;a,b,ta,tb)\tilde{\Delta}_K^{(n)}(z;t;\pm\sqrt{t},a,b)dT.$$

vanishes if λ is not an even partition (i.e., $\lambda\neq 2\mu$ for any μ). If λ is an even partition, the integral is equal to

$$\frac{(\sqrt{t})^{|\lambda|}}{(1+t)^{l(\lambda)}}\frac{N_\lambda(t;\pm\sqrt{t},a,b)v_{\lambda+}(t;\pm\sqrt{t},a,b)}{v_{\lambda+}(t^2;a,b,ta,tb)}.$$

Proof. We have

$$\begin{aligned}
&\int_T K_\lambda(z_1,\dots,z_n;t^2;a,b;a,b,ta,tb)\tilde{\Delta}_K^{(n)}(z;t;\pm\sqrt{t},a,b)dT.\\
&\quad=\frac{1}{v_\lambda(t^2;a,b;a,b,ta,tb)}\sum_{w\in B_n}\int_T R_{\lambda,w}^{(n)}(z;t^2;a,b;a,b,ta,tb)\tilde{\Delta}_K^{(n)}(z;t;\pm\sqrt{t},a,b)dT\\
&\quad\quad=\frac{2^n n!}{v_\lambda(t^2;a,b;a,b,ta,tb)}\int_T R_{\lambda,\mathrm{id}}^{(n)}(z;t^2;a,b;a,b,ta,tb)\tilde{\Delta}_K^{(n)}(z;t;\pm\sqrt{t},a,b)dT,
\end{aligned}$$

where in the last equation we have used the symmetry of the integral. We assume $\lambda_1>0$ so that $\lambda\neq 0^n$. Next, we restrict to terms involving z_1 in the integrand, and integrate with respect to z_1. Doing this computation gives the following:

$$\begin{aligned}
&\int_{T_1} z_1^{\lambda_1}\frac{(1-az_1^{-1})(1-bz_1^{-1})(1-taz_1^{-1})(1-tbz_1^{-1})}{(1-z_1^{-2})}\frac{(1-z_1^{\pm1})}{(1+\sqrt{t}z_1^{\pm1})(1-\sqrt{t}z_1^{\pm1})(1-az_1^{\pm1})(1-bz_1^{\pm1})}\\
&\quad\times\prod_{j>1}\frac{(1-t^2z_1^{-1}z_j)(1-t^2z_1^{-1}z_j^{-1})}{(1-z_1^{-1}z_j)(1-z_1^{-1}z_j^{-1})}\prod_{j>1}\frac{(1-z_1^{\pm1}z_j^{\pm1})}{(1-tz_1^{\pm1}z_j^{\pm1})}dT\\
&=\frac{1}{2\pi i}\int_C z_1^{\lambda_1-1}\frac{(z_1-ta)(z_1-tb)(1-z_1^2)}{(1-tz_1^2)(z_1+\sqrt{t})(z_1-\sqrt{t})(1-az_1)(1-bz_1)}\\
&\quad\times\prod_{j>1}\frac{(z_1-t^2z_j)(z_1-t^2z_j^{-1})(1-z_1z_j)(1-z_1z_j^{-1})}{(z_1-tz_j)(z_1-tz_j^{-1})(1-tz_1z_j)(1-tz_1z_j^{-1})}dz_1
\end{aligned}$$

Note that this integral has poles at $z_1=\pm\sqrt{t}$ and $z_1=tz_j,tz_j^{-1}$ for each $j>1$.

We first compute the residue at $z_1=\sqrt{t}$:

$$\begin{aligned}
&(\sqrt{t})^{\lambda_1-1}\frac{(\sqrt{t}-ta)(\sqrt{t}-tb)(1-t)}{(1-t^2)2\sqrt{t}(1-a\sqrt{t})(1-b\sqrt{t})}\prod_{j>1}\frac{(\sqrt{t}-t^2z_j)(\sqrt{t}-t^2z_j^{-1})(1-\sqrt{t}z_j)(1-\sqrt{t}z_j^{-1})}{(\sqrt{t}-tz_j)(\sqrt{t}-tz_j^{-1})(1-t\sqrt{t}z_j)(1-t\sqrt{t}z_j^{-1})}\\
&\quad=(\sqrt{t})^{\lambda_1}\frac{1}{2(1+t)}\prod_{j>1}\frac{(1-t\sqrt{t}z_j)(1-t\sqrt{t}z_j^{-1})(1-\sqrt{t}z_j)(1-\sqrt{t}z_j^{-1})}{(1-\sqrt{t}z_j)(1-\sqrt{t}z_j^{-1})(1-t\sqrt{t}z_j)(1-t\sqrt{t}z_j^{-1})}=\frac{(\sqrt{t})^{\lambda_1}}{2(1+t)}
\end{aligned}$$

Similarly, we can compute the residue at $z_1=-\sqrt{t}$:

$$\begin{aligned}
&(-\sqrt{t})^{\lambda_1-1}\frac{(-\sqrt{t}-ta)(-\sqrt{t}-tb)(1-t)}{(1-t^2)(-2\sqrt{t})(1+a\sqrt{t})(1+b\sqrt{t})}\prod_{j>1}\frac{(-\sqrt{t}-t^2z_j)(-\sqrt{t}-t^2z_j^{-1})(1+\sqrt{t}z_j)(1+\sqrt{t}z_j^{-1})}{(-\sqrt{t}-tz_j)(-\sqrt{t}-tz_j^{-1})(1+t\sqrt{t}z_j)(1+t\sqrt{t}z_j^{-1})}\\
&\quad=(-\sqrt{t})^{\lambda_1}\frac{1}{2(1+t)}\prod_{j>1}\frac{(1+t\sqrt{t}z_j)(1+t\sqrt{t}z_j^{-1})(1+\sqrt{t}z_j)(1+\sqrt{t}z_j^{-1})}{(1+\sqrt{t}z_j)(1+\sqrt{t}z_j^{-1})(1+t\sqrt{t}z_j)(1+t\sqrt{t}z_j^{-1})}=\frac{(-\sqrt{t})^{\lambda_1}}{2(1+t)}
\end{aligned}$$

The residues at tz_j, tz_j^{-1} can be computed in a similar manner. One can then combine these residues (at tz_j, tz_j^{-1}) with the terms from the original integrand and integrate with respect to z_j. Some computations show the resulting integral is zero; the argument is similar that used in [13, Theorem 23].

Finally, we add the residues at $z_1 = \pm\sqrt{t}$ to get

$$\frac{(\sqrt{t})^{\lambda_1}}{2(1+t)} + \frac{(-\sqrt{t})^{\lambda_1}}{2(1+t)} = \begin{cases} \frac{(\sqrt{t})^{\lambda_1}}{(1+t)}, & \text{if } \lambda_1 \text{ is even} \\ 0, & \text{if } \lambda_1 \text{ is odd.} \end{cases}$$

Thus,

$$\begin{aligned} &2^n n! \int_T R^{(n)}_{\lambda,\mathrm{id}}(z;t^2;a,b;a,b,ta,tb)\tilde{\Delta}^{(n)}_K(z;t;\pm\sqrt{t},a,b)dT \\ &\quad = \begin{cases} \frac{(\sqrt{t})^{\lambda_1}}{(1+t)} 2^{n-1}(n-1)! \int_T R^{(n-1)}_{\widehat{\lambda},\widehat{\mathrm{id}}}(z;t^2;a,b;a,b,ta,tb)\tilde{\Delta}^{(n-1)}_K(z;t;\pm\sqrt{t},a,b)dT, & \text{if } \lambda_1 \text{ is even} \\ 0, & \text{otherwise,} \end{cases} \end{aligned}$$

where $\widehat{\lambda}$ is the partition λ with the part λ_1 deleted, and $\widehat{\mathrm{id}}$ is the permutation id with z_1 deleted and signs preserved.

Consequently, the entire integral vanishes if any part is odd and if λ is even, it is equal to

$$\begin{aligned} &\frac{2^n n!}{v_\lambda(t^2;a,b;a,b,ta,tb)} \int_T R^{(n)}_{\lambda,\mathrm{id}}(z;t^2;a,b;a,b,ta,tb)\tilde{\Delta}^{(n)}_K(z;t;\pm\sqrt{t},a,b)dT \\ &\quad = \frac{2^{n-l(\lambda)}(n-l(\lambda)!}{v_{\lambda+}(t^2;a,b,ta,tb)v_{0^{n-l(\lambda)}}(t^2;a,b;a,b,ta,tb)} \frac{(\sqrt{t})^{|\lambda|}}{(1+t)^{l(\lambda)}} \\ &\qquad \times \int_T R^{(n-l(\lambda))}_{0^{n-l(\lambda)},\mathrm{id}}(z;t^2;a,b;a,b,ta,tb)\tilde{\Delta}^{(n-l(\lambda))}_K(z;t;\pm\sqrt{t},a,b)dT, \end{aligned}$$

where, by abuse of notation in the last line, we use id to denote the identity element in $B_{n-l(\lambda)}$. By (6), the last line is equal to

$$\begin{aligned} &\frac{2^{n-l(\lambda)}(n-l(\lambda)!}{v_{\lambda+}(t^2;a,b,ta,tb)} \frac{(\sqrt{t})^{|\lambda|}}{(1+t)^{l(\lambda)}} \int_T K^{(n-l(\lambda))}_{0^{n-l(\lambda)},\mathrm{id}}(z;t^2;a,b;a,b,ta,tb)\tilde{\Delta}^{(n-l(\lambda))}_K(z;t;\pm\sqrt{t},a,b)dT \\ &\quad = \frac{1}{v_{\lambda+}(t^2;a,b,ta,tb)} \frac{(\sqrt{t})^{|\lambda|}}{(1+t)^{l(\lambda)}} \int_T K^{(n-l(\lambda))}_{0^{n-l(\lambda)}}(z;t^2;a,b;a,b,ta,tb)\tilde{\Delta}^{(n-l(\lambda))}_K(z;t;\pm\sqrt{t},a,b)dT \\ &\quad = \frac{1}{v_{\lambda+}(t^2;a,b,ta,tb)} \frac{(\sqrt{t})^{|\lambda|}}{(1+t)^{l(\lambda)}} \int_T \tilde{\Delta}^{(n-l(\lambda))}_K(z;t;\pm\sqrt{t},a,b)dT \\ &\quad = \frac{(\sqrt{t})^{|\lambda|}}{(1+t)^{l(\lambda)}} \frac{N_\lambda(t;\pm\sqrt{t},a,b)v_{\lambda+}(t;\pm\sqrt{t},a,b)}{v_{\lambda+}(t^2;a,b,ta,tb)}, \end{aligned}$$

since $K^{(l)}_{0^l}(z;t;a,b;t_0,\dots,t_3) = 1$ by Theorem 2.6 and $n - l(\lambda) = m_0(\lambda)$.

□

3. Nonsymmetric Hall-Littlewood polynomials of type BC

3.1. **Background and Notation.** We first introduce the affine Hecke algebra of type BC, a crucial object in the study of nonsymmetric Koornwinder polynomials. We retain the notation on partitions, compositions and orderings of the previous section.

Definition 3.1. (see [9, 11, 12, 10]) The *affine Hecke algebra H of type BC* is defined to be the $\mathbb{C}(q,t,a,b,c,d)$ algebra with generators $T_0, T_1, \dots, T_n$, subject to the following braid relations

$$\begin{aligned} T_iT_j &= T_jT_i, \quad |i-j| \geq 2, \\ T_iT_jT_i &= T_jT_iT_j, \quad |i-j| = 1, i,j \neq 0,n, \\ T_iT_{i+1}T_iT_{i+1} &= T_{i+1}T_iT_{i+1}T_i \quad (i = 0, i = n-1) \end{aligned}$$

and the quadratic relations

$$\begin{aligned}(T_0+1)(T_0+cd/q)&=0,\\ (T_i+1)(T_i-t)&=0, \quad i\neq 0,n,\\ (T_n+1)(T_n+ab)&=0.\end{aligned}$$

Recall that, by the Noumi representation (see [10, 11]), there is an action of H on the vector space of Laurent polynomials $\mathbb{C}(q^{1/2},t,a,b,c,d)[x_1^{\pm 1},\dots,x_n^{\pm 1}]$ (here $x_1,\dots,x_n$ are n independent indeterminates) as follows:

$$\begin{aligned}T_0 f&=-(cd/q)f+\frac{(1-c/x_1)(1-d/x_1)}{1-q/x_1^2}(f^{s_0}-f)\\ T_i f&=tf+\frac{x_{i+1}-tx_i}{x_{i+1}-x_i}(f^{s_i}-f) \quad \text{(for } 0<i<n)\\ T_n f&=-abf+\frac{(1-ax_n)(1-bx_n)}{1-x_n^2}(f^{s_n}-f),\end{aligned}$$

where $f^{s_0}(x_1,\dots,x_n)=f(q/x_1,x_2,\dots,x_n)$, $f^{s_i}(x_1,\dots,x_n)=f(x_1,\dots,x_{i-1}x_{i+1}x_ix_{i+2},\dots,x_n)$ for $0<i<n$, and $f^{s_n}(x_1,\dots,x_n)=f(x_1,\dots,x_{n-1},1/x_n)$. Note that, for $0<i\leq n$, the action of T_i on polynomials is independent of q; this will be crucial for the rest of the paper.

We will denote the nonsymmetric Koornwinder polynomials in n variables by $U_\lambda^{(n)}(x;q,t;a,b,c,d)$ [7, 10, 11, 12]. Recall that these polynomials are indexed by compositions λ. We will remind the reader how these polynomials are defined, but we must first set up some relevant notation. Let $^\iota$ be the involution defined by

$$^\iota: q\to q^{-1}, t\to t^{-1}, a\to a^{-1}, b\to b^{-1}, c\to c^{-1}, d\to d^{-1}, z^\mu\to z^\mu$$

and let $^-$ be the involution defined by

$$^-: q\to q, t\to t, a\to a, b\to b, c\to c, d\to d, z^\mu\to z^{-\mu}.$$

Define the weight

$$\Delta_K^{(n)}(z;q,t;a,b,c,d)=\prod_{1\leq i\leq n}\frac{(z_i^2,qz_i^{-2};q)}{(az_i,bz_i,cz_i,dz_i,aqz_i^{-1},bqz_i^{-1},cz_i^{-1},dz_i^{-1};q)} \times\prod_{1\leq i<j\leq n}\frac{(z_iz_j^{\pm 1},qz_i^{-1}z_j^{\pm 1};q)}{(tz_iz_j^{\pm 1},qtz_i^{-1}z_j^{\pm 1};q)}, \tag{16}$$

i.e., the full nonsymmetric density, see [9]. As in the symmetric case, when the parameters are clear from context, we will suppress them to make the notation easier. For example, we will simply write $\Delta_K^{(n)}$ for this weight. Note the following formula for the nonsymmetric density at the specialization $q=0$:

$$\prod_{1\leq i\leq n}\frac{(1-z_i^2)}{(1-az_i)(1-bz_i)(1-cz_i)(1-dz_i)(1-cz_i^{-1})(1-dz_i^{-1})}\prod_{1\leq i<j\leq n}\frac{1-z_iz_j^{\pm 1}}{1-tz_iz_j^{\pm 1}}. \tag{17}$$

We will write $\Delta_K^{(n)}(z;t;a,b,c,d)$ to indicate this particular limiting case. Note also that the specialization of $\Delta_K^{(n)}(z;q,t;a,b,c,d)$ given by $t=1, a=1, b=-1, c=d=0$, which is independent of q, gives $\Delta_K=1$.

With this terminology, consider the following inner product on functions of n variables with parameters q,t,a,b,c,d:

$$\langle f,g\rangle_q=\int_T f\bar{g}^\iota\Delta_K^{(n)}(z;q,t;a,b,c,d)dT.$$

Note that it is the constant term of $f\bar{g}^\iota\Delta_K$ (the coefficient on z^0). Also, denote by $\langle,\rangle_0$ the following inner product:

$$\langle f,g\rangle_0=\int_T f\bar{g}^\iota\Delta_K^{(n)}(z;t;a,b,c,d)dT, \tag{18}$$

involving the $q=0$ degeneration of the full nonsymmetric Koornwinder weight as in (17).

Recall that the polynomials $\{U_\mu^{(n)}(x;q,t;a,b,c,d)\}_{\mu\in\mathbb{Z}^n}$ are uniquely defined by the following conditions:

$$(i)\ \ U_\mu = x^\mu + \sum_{\nu\prec\mu} w_{\mu\nu}x^\nu$$

$$(ii)\ \ \langle U_\mu, x^\nu\rangle_q = 0 \text{ if } \nu\prec\mu,$$

where we write x^μ for the monomial $x_1^{\mu_1}x_2^{\mu_2}\cdots x_n^{\mu_n}$.

Definition 3.2. For a partition λ with $l(\lambda)\le n$, define

$$E_\lambda^{(n)}(z;c,d) = E_\lambda(z_1,\ldots,z_n;c,d) = \prod_{\lambda_i>0} z_i^{\lambda_i}(1-cz_i^{-1})(1-dz_i^{-1}).$$

3.2. **Main Results.** Our first goal will be show that, under the assumption that λ is a partition with $l(\lambda)\le n$, $E_\lambda^{(n)}(z;c,d)$ is the $q=0$ limiting case of the nonsymmetric Koornwinder polynomial $U_\lambda^{(n)}(x;q,t;a,b,c,d)$.

Theorem 3.3. *(Triangularity) The polynomials $E_\lambda^{(n)}(z;c,d)$ are triangular with respect to dominance ordering, i.e.,*

$$E_\lambda^{(n)}(z;c,d) = z^\lambda + \sum_{\mu\prec\lambda} c_\mu z^\mu$$

for all partitions λ.

Proof. It is clear that $E_\lambda^{(n)}(z;c,d) = z^\lambda + (\text{dominated terms})$, since the term inside the product definition of $E_\lambda^{(n)}(z;c,d)$ is $z_i^{\lambda_i} - (c+d)z_i^{\lambda_i-1} + cdz_i^{\lambda_i-2}$. □

Theorem 3.4. *We have the following constant term evaluation in the nonsymmetric case (with respect to $q=0$ limit of the nonsymmetric density as in (17))*

$$\int_T \Delta_K^{(n)}(z;t;a,b,c,d)dT = \prod_{i=0}^{n-1}\frac{1}{(1-t^iac)(1-t^ibc)(1-t^icd)(1-t^iad)(1-t^ibd)}\prod_{j=n-1}^{2n-2}(1-t^jabcd).$$

Proof. This follows from the proof of Theorem 2.7, in particular recall (14). □

Theorem 3.5. *(Orthogonality) Let λ be a partition with $l(\lambda)\le n$ and $\mu\in\mathbb{Z}^n$ a composition, such that $\mu\prec\lambda$. Then we have $\langle E_\lambda^{(n)}(z;c,d), z^\mu\rangle_0 = 0$.*

Proof. Fix λ a partition. First note that, by definition of the inner product $\langle\cdot,\cdot,\rangle_0$ in (18) we have

$$\langle E_\lambda^{(n)}(z;c,d), z^\mu\rangle_0$$
$$= \int_T E_\lambda^{(n)}(z;c,d)z^{-\mu}\prod_{1\le i\le n}\frac{(1-z_i^2)}{(1-az_i)(1-bz_i)(1-cz_i)(1-dz_i)(1-cz_i^{-1})(1-dz_i^{-1})}\prod_{1\le i<j\le n}\frac{1-z_iz_j^{\pm1}}{1-tz_iz_j^{\pm1}}.$$

We will first show $\langle E_\lambda^{(n)}(z;c,d), z^\mu\rangle_0 = 0$ for all compositions μ satisfying the following two properties:

Condition (i) $\mu \overset{\text{lex}}{<} \lambda$, so in particular there exists $1\le i\le n$ such that $\mu_1=\lambda_1,\ldots,\mu_{i-1}=\lambda_{i-1}$ and $\mu_i<\lambda_i$.

Condition (ii) $\lambda_i\ne 0$ (where i is as in (i)).

We mention that condition (ii) is necessary because of the difference between nonzero and zero parts in Definition 3.2; in particular if $\lambda_i=0$ then one does not have the term $z_i^{\lambda_i}(1-cz_i^{-1})(1-dz_i^{-1})$ in $E_\lambda^{(n)}(z;c,d)$ (so that one still has the terms $1/(1-cz_i^{-1})(1-dz_i^{-1})$ in the product $E_\lambda^{(n)}(z;c,d)\Delta_K^{(n)}$) . We give a proof by induction on n, the number of variables. Note first that condition (ii) implies that $\lambda_1,\ldots,\lambda_{i-1}\ne 0$. Consider the case $n=1$. Then in particular $i=1$ and conditions (i) and (ii) give $\mu_1<\lambda_1\ne 0$. One can then compute

$$\langle E_\lambda^{(n)}, z^\mu\rangle_0 = \int_T z_1^{\lambda_1-\mu_1}\frac{(1-z_1^2)}{(1-az_1)(1-bz_1)(1-cz_1)(1-dz_1)}dT,$$

since $\lambda_1-\mu_1>0$ this is necessarily zero. Now suppose the claim holds for $n-1$, we show it holds for n.

We may restrict the n-dimensional integral $\langle E_\lambda(z), z^\mu\rangle_0$ to the contribution involving z_1, one computes it to be

$$\int_T z_1^{\lambda_1-\mu_1} \frac{(1-z_1^2)}{(1-az_1)(1-bz_1)(1-cz_1)(1-dz_1)} \prod_{j>1} \frac{1-z_1 z_j^{\pm 1}}{1-tz_1 z_j^{\pm 1}} dT.$$

If $i=1$, then $\lambda_1 > \mu_1$ and this integral (and consequently the n-dimensional integral) are zero. If $i>1$, then $\lambda_1 = \mu_1$ and this integral is 1. In this case, one notes that the resulting $n-1$ dimensional integral is exactly:

$$\int_T E_{\widehat{\lambda}}^{(n-1)}(z_2,\dots,z_n) z^{-\widehat{\mu}} \Delta_K^{(n-1)} dT,$$

where $\widehat{\lambda} = (\lambda_2,\dots,\lambda_n)$ and $\widehat{\mu} = (\mu_2,\dots,\mu_n)$. Note that conditions (i) and (ii) hold for $\widehat{\mu}$ and $\widehat{\lambda}$, and since this is the $n-1$ variable case we may appeal to the induction hypothesis. Thus, the above integral is zero; consequently $\langle E_\lambda^{(n)}, z^\mu\rangle = 0$ as desired.

Finally, it remains to show that $\mu \prec \lambda$ implies conditions (i) and (ii). Recall that there are two cases for $\mu \prec \lambda$. In case 1), note that we have $\mu \le \mu^+ < \lambda$ with respect to the dominance ordering, so $\mu < \lambda$. This implies $\mu \overset{\text{lex}}{<} \lambda$ by Lemma 2.3. In case 2), it is clear. Now we show condition (ii). Suppose for contradiction that $\lambda_i = 0$, so that $\mu_1 = \lambda_1 > 0, \dots, \mu_{i-1} = \lambda_{i-1} > 0$ and $\mu_i < \lambda_i = 0$ and $\lambda_k = 0$ for all $i < k \le n$. Then note that $\sum_{k=1}^i (\mu^+)_k > \sum_{k=1}^i \lambda_k$, which contradicts $\mu^+ \le \lambda$. Thus, we must have $\lambda_i \neq 0$ as desired. □

Theorem 3.6. *Let λ be a partition with $l(\lambda) \le n$, then*

$$\begin{aligned}\langle E_\lambda^{(n)}(z;c,d), E_\lambda^{(n)}(z;c,d)\rangle_0 &= \langle E_\lambda^{(n)}(z;c,d), z^\lambda\rangle_0 \\ &= \prod_{i=0}^{m_0(\lambda)-1} \frac{1}{(1-t^iac)(1-t^ibc)(1-t^icd)(1-t^iad)(1-t^ibd)} \prod_{j=m_0(\lambda)-1}^{2m_0(\lambda)-2} (1-t^j abcd)\end{aligned}$$

Proof. The first equality follows from Theorems 3.3 and 3.5. For the second equality, we use arguments similar to those used in the proof of Theorem 3.5. We first note that

$$\begin{aligned}&\langle E_\lambda^{(n)}(z;c,d), z^\lambda\rangle_0 \\ &= \int_T E_\lambda^{(n)}(z;c,d) z^{-\lambda} \prod_{1\le i\le n} \frac{(1-z_i^2)}{(1-az_i)(1-bz_i)(1-cz_i)(1-dz_i)(1-cz_i^{-1})(1-dz_i^{-1})} \prod_{1\le i<j\le n} \frac{1-z_iz_j^{\pm 1}}{1-tz_iz_j^{\pm 1}}.\end{aligned}$$

One can integrate with respect to z_1, holding the remaining variables fixed; the reader can verify that the result is $\langle E_{\widehat{\lambda}}^{(n-1)}(z;c,d), z^{\widehat{\lambda}}\rangle_0$, where $\widehat{\lambda} = (\lambda_2,\dots,\lambda_n)$. Iterating this argument shows that

$$\langle E_\lambda^{(n)}(z;c,d), z^\lambda\rangle_0 = \int_T \Delta_K^{(m_0(\lambda))} dT.$$

By Theorem 3.4, this is equal to

$$\prod_{i=0}^{m_0(\lambda)-1} \frac{1}{(1-t^iac)(1-t^ibc)(1-t^icd)(1-t^iad)(1-t^ibd)} \prod_{j=m_0(\lambda)-1}^{2m_0(\lambda)-2} (1-t^j abcd),$$

as desired. □

In the next theorem, we will prove that these polynomials $E_\lambda^{(n)}(z;c,d)$ are indeed the nonsymmetric Koornwinder polynomials indexed by a partition in the limit $q \to 0$. In order to do this, we will show that the limit is actually well-defined, and that polynomials satisfying the above triangularity and orthogonality conditions are uniquely determined.

Theorem 3.7. *The $q \to 0$ limit of the nonsymmetric Koornwinder polynomials are well-defined. Moreover, for a partition λ, $\lim_{q\to 0} U_\lambda^{(n)}(x;q,t;a,b,c,d) = E_\lambda^{(n)}(z;c,d)$.*

Proof. By virtue of the triangularity condition for nonsymmetric Koornwinder polynomials, we can write $U_\lambda^{(n)}(x;q,t;a,b,c,d) = x^\lambda + \sum_{\mu\prec\lambda} c_\mu x^\mu$. The orthogonality condition for these polynomials then gives

$$\langle U_\lambda^{(n)}(x;q,t;a,b,c,d), x^\nu\rangle_q = \Big\langle x^\lambda + \sum_{\mu\prec\lambda} c_\mu x^\mu, x^\nu\Big\rangle_q = 0$$

for all $\nu \prec \lambda$. By linearity of the inner product, one can rewrite this as

$$\Big\langle \sum_{\mu\prec\lambda} c_\mu x^\mu, x^\nu\Big\rangle_q = -\langle x^\lambda, x^\nu\rangle_q$$

for all $\nu \prec \lambda$. So the coefficient vector $\vec{c}$ is uniquely determined by the equation

$$\vec{c}A_{[\prec\lambda]} = -\vec{A}_\lambda, \tag{19}$$

where $A_{[\prec\lambda]}$ is the inner product matrix of the monomials $\{x^\nu\}_{\nu\prec\lambda}$, and $\vec{A}_\lambda$ is the column vector of inner products of x^λ with $\{x^\nu\}_{\nu\prec\lambda}$. We need to check that the limit as $q\to 0$ of the entries of $\vec{c}$ is well-defined. It suffices to show that $\lim_{q\to 0}\det A_{[\prec\lambda]}$ is not equal to zero. We will exhibit a specialization of the parameters $(t, t_0, \ldots, t_3)$ for which this limit is nonzero. To this end, consider the following specialization of parameters: $(t = 1, t_0 = 1, t_1 = -1, t_2 = t_3 = 0)$; note that it is independent of q. Under this specialization, the inner product weight becomes $\Delta_K = 1$, and the inner product for monomials is $\langle x^\lambda, x^\mu\rangle_q = \delta_{\lambda,\mu}$. In particular, the matrix $A_{[<\lambda]}$ is the identity matrix and it has determinant equal to one. We write $\lim_{q\to 0}\vec{c}$ for the $q\to 0$ limit of the entries of the vector $\vec{c}$.

Now, write $E_\lambda^{(n)}(x;c,d) = x^\lambda + \sum_{\mu\prec\lambda} d_\mu x^\mu$, using Theorem 3.3. Then, using Theorem 3.5, exactly as in the previous paragraph we find $\vec{d}$ satisfies $\vec{d}A^0_{[<\lambda]} = -A^0_\lambda$, where $\cdot^0$ denotes the specialization of the inner product for monomials at $q\to 0$. But since the vectors $\vec{c}$ and $\vec{d}$ are uniquely determined via the method discussed above, and the matrices here are the $q\to 0$ specializations of the ones in (19), we must have $\lim_{q\to 0}\vec{c} = \vec{d}$, as desired. $\square$

In the next proposition, we will show that one can recursively construct all the nonsymmetric Koornwinder polynomials at $q = 0$ starting from the ones indexed by partitions, i.e, the $\{E_\lambda^{(n)}(z;c,d)\}_{\lambda\in\Lambda_n^+}$ as defined in Definition 3.2. We will follow the notation in [11, 12]. In translating between the multiplicity function there and our parameters $\{a,b,c,d\}$ one can take the following reparametrization:

$$\{a,b,c,d\} = \{t_0\check{t_0}q^{1/2}, -t_0\check{t_0}^{-1}q^{1/2}, t_n\check{t_n}, -t_n\check{t_n}^{-1}\}$$

Proposition 3.8. *Let $i\in\{1,\cdots,n\}$ and let $\lambda\in\Lambda$. Then the equation*

$$T_iU_\lambda = \zeta_i(\gamma_\lambda)U_\lambda + \eta_i(\gamma_\lambda)U_{s_i\lambda}, \tag{20}$$

where the parameters $\zeta_i(\gamma_\lambda)$ and $\eta_i(\gamma_\lambda)$ are as in [12] Proposition 6.1, admits the limit $q\to 0$.

Proof. We freely use the notation in [12]. One first notes that the T_i (for $1\le i\le n$) do not depend on q. There are three cases depending on $\langle\lambda,a_i\rangle$: $\langle\lambda,a_i\rangle = 0$, $\langle\lambda,a_i\rangle < 0$ and $\langle\lambda,a_i\rangle > 0$. If $\langle\lambda,a_i\rangle = 0$, (20) becomes $T_iU_\lambda = t_iU_\lambda$. If $\langle\lambda,a_i\rangle < 0$, (20) becomes $T_iU_\lambda = \zeta_i(\gamma_\lambda)U_\lambda + t_iU_{s_i\lambda}$, so it suffices to show $\zeta_i(\gamma_\lambda)$ is well-defined in the limit $q\to 0$. One can compute

$$\zeta_i(\gamma_\lambda) = \frac{c_1q^{\langle\lambda,a_i\rangle}\left[\tilde{t}_{a_i}^{-1} - \tilde{t}_{a_i}\right] + c_2q^{\langle\lambda,a_i\rangle/2}\left[\tilde{t}_{a_i/2}^{-1} - \tilde{t}_{a_i/2}\right]}{1 - c_3q^{\langle\lambda,a_i\rangle}} = \frac{c_1\left[\tilde{t}_{a_i}^{-1} - \tilde{t}_{a_i}\right] + c_2q^{-\langle\lambda,a_i\rangle/2}\left[\tilde{t}_{a_i/2}^{-1} - \tilde{t}_{a_i/2}\right]}{q^{-\langle\lambda,a_i\rangle} - c_3},$$

for some constants c_1, c_2, c_3 not depending on q. Taking the limit $q\to 0$ of this last expression gives

$$\frac{c_1\left[\tilde{t}_{a_i}^{-1} - \tilde{t}_{a_i}\right]}{-c_3},$$

so the above equation becomes

$$U_{s_i\lambda} = \frac{\left[T_i + \frac{c_1\left[\tilde{t}_{a_i}^{-1} - \tilde{t}_{a_i}\right]}{c_3}\right]}{t_i}U_\lambda.$$

Finally, the case $\langle\lambda,a_i\rangle > 0$ remains. In this case, (20) becomes

$$T_iU_\lambda = \zeta_i(\gamma_\lambda)U_\lambda + \tilde{t}_i^{-3}\tilde{\nu}_{a_i}(\gamma_\lambda)\tilde{\nu}_{-a_i}(\gamma_\lambda)U_{s_i\lambda}.$$

Similar reasoning to the previous case shows that $\lim_{q\to 0} \zeta_i(\gamma_\lambda) = 0$. A limit computation identical to the previous case shows that

$$\lim_{q\to 0} \tilde{\nu}_{a_i}(\gamma_\lambda)\tilde{\nu}_{-a_i}(\gamma_\lambda) = \tilde{t}_i^2,$$

so ultimately we obtain the equation

$$U_{s_i\lambda} = -\tilde{t}_i T_i U_\lambda.$$

□

Definition 3.9. Define $E_\mu^{(n)}$ recursively for μ a composition using the previous proposition and Definition 3.2, i.e., for $1 \le i \le n$

$$E_{s_i\lambda}^{(n)} := \lim_{q\to 0} \left[\frac{T_i - \zeta_i(\gamma_\lambda)}{\eta_i(\gamma_\lambda)} \right] E_\lambda^{(n)}.$$

This definition, along with Definition 3.2, provides a complete construction for the $q = 0$ limiting case of the nonsymmetric Koornwinder polynomials. As a byproduct of the full orthogonality for the $\{U_\lambda^{(n)}(x; q, t; a, b, c, d)\}$ we obtain the full orthogonality for the $q = 0$ limiting case:

Corollary 3.10. *Let $\lambda, \mu \in \mathbb{Z}^n$ be compositions, such that $\mu \prec \lambda$. Then we have $\langle E_\lambda^{(n)}, z^\mu \rangle_0 = 0$.*

Proof. This follows from the orthogonality for the full nonsymmetric Koornwinder polynomials, along with Theorem 3.7 and Definition 3.9. □

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